Journal of Sound and Vibration (1997) **203**(5), 731–743

THE NON-LINEAR DYNAMIC RESPONSE OF PAIRED CENTRIFUGAL PENDULUM VIBRATION ABSORBERS

C.-T. LEE AND S. W. SHAW

Department of Mechanical Engineering, *Michigan State University*, *East Lansing*, *MI* 48824, *U.S.A.*

(*Received* 20 *October* 1995, *and in final form* 22 *July* 1996)

The dynamic response of a pair of identical centrifugal pendulum vibration absorbers is considered. Of particular interest here is the effectiveness of using non-linear behavior to simultaneously reduce torsional vibrations in rotating machines that arise from external torques consisting of multiple harmonics. The equations of motion for the coupled absorber/rotary system are given in which the absorbers are allowed to undergo finite amplitude motions. The Method of Multiple Scales (MMS) is applied to the second order to achieve approximate steady-state solutions of these equations. It is found that the pair of absorbers are capable of simultaneously cancelling two harmonics when the absorber damping is kept small, although higher order harmonics may be amplified. This is achieved by a bifurcation of the unison motion of the absorbers to a motion with a relative phase shift and an amplitude difference. Due to this bifurcation, the performance of the absorber pair is superior to that of a single absorber having the same total inertia. This study focuses on the analytical aspects of the problem and simulation verification of the results.

1. INTRODUCTION

Centrifugal pendulum vibration absorbers (CPVA's) are used to reduce vibrations in rotating and reciprocating machinery. They have been successfully employed in internal combustion engines and helicopter rotors to counteract the oscillating components of applied forces and torques of a given harmonic order over a range of operating speeds (for applications, see references [1–4]). Due to spatial and balancing considerations, the practical implementation of CPVA's usually requires a number of individual absorbers stationed about the center of rotation. When designing absorber systems for attenuating torsional vibrations, sizing estimates and analytical performance evaluation predictions are based on treating the absorbers as if they all move in exact unison, acting essentially as a single absorber [1, 5]. An exception is the recently proposed "sub-harmonic" pair of CPVA's that uses an "out-of-phase" motion of paired identical absorbers [6].

The current study is motivated by observations made during numerical simulations of a rotary system with multiple CPVA's, where it was observed that the unison motion undergoes a dynamic instability that results in the absorbers moving with relative phase shifts and amplitude differences [7]. Recently, an approximate stability criterion for the unison motion for the case of N identical tautochronic absorbers and a purely harmonic torque has been obtained [8]. The present study is limited to the case with two identical CPVA's, but goes further in that the applied torque is taken to be multi-harmonic and the post-critical response is determined. It is observed in the present simulations that the

0022–460X/97/250731 + 13 $25.00/0/sv960707

CPVA's move in unison at small torque levels, but when the torque amplitude exceeds a certain value they undergo a bifurcation in which the absorber motions deviate from one another. Furthermore, it is observed that this bifurcation has a generally beneficial effect on the overall torsional vibration level of the system.

The goal of this study is to develop an analytical formulation that is capable of predicting the parameter conditions associated with the instability of the unison motion and the performance of the absorber system beyond the instability.

The system under investigation is modelled as a rotating disk with two identical absorbers with identical paths, where the disk is subjected to an external torque consisting of two harmonics. The absorbers are assumed to be designed such that their centers of mass follow a particular epicycloidal path that provides an approximately *tautochronic* motion of the absorbers [5]. This path shape helps to avoid the potentially disastrous effects of non-linear mistuning [9, 10]. (Note that most applications use cycloidal paths or simple circular paths with small intentional mistuning. These partially account for amplitude-induced mistuning, whereas epicycloidal paths are optimal in this regard; see reference [5].) The perturbation analysis is carried out by at second order Method of Multiple Scales (MMS) technique recently proposed by Lee and Lee [11], which is a simplified version of the MMS offered by Rahman and Burton [12]. This approach captures the desired results.

This paper is arranged as follows. Section 2 describes the basic system in terms of assumptions and a dynamical model. Section 3 contains a summary of the perturbation analysis, the main results and a comparison with numerical simulations. Since the system possesses many parameters, a few special cases of interest are selected for consideration. Section 4 closes the paper with some conclusions, design suggestions, and a conjecture regarding systems with more than two absorbers.

2. THE BASIC SYSTEM

The model of the general system consists of a disk which is free to rotate in a plane about a fixed point O and N absorber masses which move along epicycloidal paths relative to the disk. Figure 1 shows the case for $N = 1$. Of particular interest in the present work is the case $N = 2$. The disk respresents the inertia of the primary system, and its angular orientation is denoted by θ which is measured relative to an inertial frame of reference. It has a moment of inertia of I_d respect to point O. The ith absorber is modelled as a point mass m_i, riding on an epicycloidal path tuned to order n_i, which is specified by

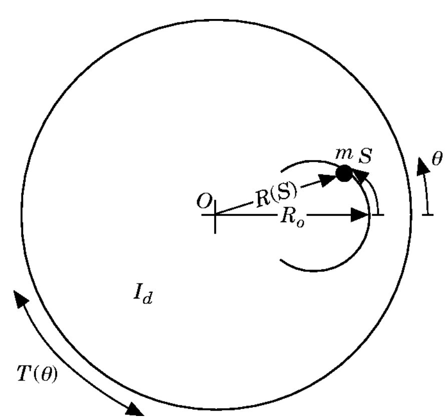

Figure 1. Schematic diagram of the basic system with one absorber.

$R_i^2 = R_{0i}^2 - n_i^2 S_i^2$, where R_i denotes the distance from a point on the ith absorber path to point O, R_{0i} is the value of R_i at the vertex point of the path, and S_i is an arc length variable along the path. The variable S_i is also used to denote the location of m_i during its motion, and it is chosen such that $S_i = 0$ at the vertex. The absorber has a nominal moment of inertia $I_i = m_i R_{0i}^2$ at its vertex with respect to point O. Note that in order to generalize to the case of non-point-mass absorbers, the moments of inertia about their own centers of mass can be included in I_d.

The model also includes linear viscous-type damping for the disk and each absorber, with c_0 and c_{ai} as the corresponding damping coefficients. The external torque is assumed to be a combination of a constant component, T_0, and an oscillating components, $T(\theta)$. (These include all external torques, including any loads acting). The constant torque T_0 balances the mean component of the torque which arises from bearing damping at point O, thus setting the mean rotation speed of the disk, Ω, while the oscillating torque $T(\theta)$ is the source of speed fluctuations and torsional vibration.

Assuming that gravitational effects are small and the associated potential energy is therefore negligible, the governing equations of motion for this system are determined by applying Lagrange's method to the kinetic energy. After non-dimensionalization the equations of motion are given by

$$\ddot{s}_i + g_i(s_i)\dot{z} + n_i^2 s_i (1+z)^2 = -\hat{\mu}_{ai}\dot{s}_i, \qquad i = 1, \ldots, N, \tag{1}$$

$$\sum_{k=1}^{N} b_k \left[-2n_k^2 s_k \dot{s}_k (1+z) + (1 - n_k^2 s_k^2)\dot{z} + g_k(s_k)\ddot{s}_k + \frac{\mathrm{d}g_k}{\mathrm{d}s_k}(s_k)\dot{s}_k^2 \right] + \dot{z}$$

$$= \sum_{k=1}^{N} b_k \hat{\mu}_{ak} g_k(s_k)\dot{s}_k - \hat{\mu}_0(1+z) + \hat{\Gamma}_0 + \hat{\Gamma}(\theta), \tag{2}$$

where $(\dot{\cdot})$ denotes $\mathrm{d}(\cdot)/\mathrm{d}\tau$, $\tau = \Omega t$, $s_i = S_i/R_{0i}$, $z = (1/\Omega)(\mathrm{d}\theta/\mathrm{d}t) - 1 = \mathrm{d}\dot{\theta}/\mathrm{d}\tau - 1$, $b_i = I_i/I_d$, $\hat{\mu}_{ai} = c_{ai}/m_i\Omega$, $\hat{\mu}_0 = c_0/I_d\Omega$, $\hat{\Gamma}_0 = T_0/I_d\Omega^2$, $\hat{\Gamma}(\theta) = T(\theta)/I_d\Omega^2$, and

$$g_i(s_i) = \sqrt{1 - n_i^2(n_i^2+1)s_i^2}. \tag{3}$$

Equation (1) represents the dynamics of the absorber masses, while equation (2) is a dynamic torque balance for the disk. It should be noted that the $\hat{\Gamma}$'s represent the torque amplitudes, the $\hat{\mu}$'s represent damping coefficients, b_i is the ratio of the ith absorber inertia to the disk inertia (a quantity typically much less than unity), s_i's denote absorber displacements, z represents the deviation of the non-dimensionalized angular speed from unity, and τ is a time scale based on the nominal angular speed, Ω. Herein the case of two absorbers, $N = 2$, with identical paths is considered.

Two consecutive changes of variables are applied to the equations of motion in order to transform them into a more convenient form for analysis. The first one transforms the independent variable from τ to θ. The advantage offered by this is that the large non-linear term $\hat{\Gamma}(\theta)$ can be converted into an external, periodic excitation. The second transformation decouples the undamped version of the non-linear equations of motion at the linear order, and this allows one to easily identify the dynamic mode that is involved in the bifurcation.

The first transformation is accomplished by making the reasonable assumption that the system rotates with $(1+z)$ always positive, i.e., the disk never reverses its direction of rotation. This allows one to convert the equations of motion via a change of variables to a form in which θ is the independent variable. Since θ appears explicitly only in the applied torque in equation (2), the torque plays the role of an external periodic excitation in the new equations of motion. The transformation is realized by expressing τ as a function of θ (the solution for this relationship is never actually required in the analysis) and re-expressing all functions of τ as functions of θ. In terms of the new variables, the angular acceleration of the disk, $\dot{z}$, becomes $(1+z)z'$, where $(\cdot)'$ denotes $\mathrm{d}(\cdot)/\mathrm{d}\theta$. The equations of motion which result from this transformation are given by

$$s_i''(1+z) + [s_i' + g_i(s_i)]z' + n_i^2 s_i(1+z) = -\hat{\mu}_{ai} s_i', \qquad i = 1, \ldots, N, \tag{4}$$

$$\sum_{k=1}^{N} b_k(1+z)\left\{\left[-2n_k^2 s_k s_k' + g_k(s_k)s_k'' + \frac{\mathrm{d}g_k}{\mathrm{d}s_k}(s_k)s_k'^2\right](1+z) + [(1-n_k^2 s_k^2) + g_k(s_k)s_k']z'\right\}$$

$$+(1+z)z' = \sum_{k=1}^{N} b_k \hat{\mu}_{ak} g_k(s_k) s_k'(1+z) - \hat{\mu}_0(1+z) + \hat{\Gamma}_0 + \hat{\Gamma}(\theta). \tag{5}$$

Note that the differential order of the equations of motion is reduced by one with this transformation, since the original form contains $\dot{z} = \ddot{\theta}$ whereas the new version contains z', and z, not θ, is the important dependent variable. However, also note that this transformation introduces additional non-linear terms and renders the system non-autonomous in terms of θ.

When there are no applied oscillatory torques ($\hat{\Gamma}(\theta) = 0$), there exists a motion in which the disk rotates at a constant rate ($z \equiv 0$), the absorbers remain at their vertices ($s_i \equiv 0$), and the bearing torque is balanced by the constant torque ($\hat{\mu}_0 = \hat{\Gamma}_0$). Therefore, the values of the dependent variables s_1, s_2, and z can be assumed to be small when the level of the oscillatory excitation $\hat{\Gamma}(\theta)$ is small. This will be the generating motion for the perturbation analysis.

The purpose of the second transformation is to at least partially decouple the undamped version of equations (4) and (5) at the linear order. Since the number of absorbers is two, i.e., $N = 2$, and the absorbers and their paths are designed to be identical to each other, the system parameters are simplified as follows: $b_1 = b_2 \equiv b$, $\hat{\mu}_{a1} = \hat{\mu}_{a2} \equiv \hat{\mu}_a$, and $n_1 = n_2 \equiv n$. Due to a special symmetry in the $N = 2$ version of the equations of motion, it is convenient to define the following dimensionless co-ordinates which replace s_1 and s_2:

$$\xi_1 = (s_1 + s_2)/\sqrt{2}, \qquad \xi_2 = (-s_1 + s_2)/\sqrt{2}, \tag{6}$$

with its inverse transformation provided by

$$s_1 = (\xi_1 - \xi_2)/\sqrt{2}, \qquad s_2 = (\xi_1 + \xi_2)/\sqrt{2}. \tag{7}$$

This linear transformation puts the undamped linear part of the system in a convenient form. A similar transformation was used by Cronin in order to decouple the linearized equations for a set of CPVA's used for shake reduction [4].

When this transformation is applied to the equations of motion (4) and (5), one obtains

$$\xi_r'' + n^2(1+2b)\xi_1 = -(1+2b)(\hat{\mu}_a \xi_1' + \xi_1'' z + \xi_1' z' + n^2 \xi_1 z + h_1 z') - \sqrt{2}f, \tag{8}$$

$$\xi_2'' + n^2 \xi_2 = -(\hat{\mu}_a \xi_2' + \xi_2'' z + \xi_2' z' + n^2 \xi_2 z + h_2 z'), \tag{9}$$

$$(1+2b)z' + \sqrt{2}b\xi_1'' = f, \tag{10}$$

where

$$h_1 = h_1(\xi_1, \xi_2) = \frac{1}{\sqrt{2}}\left[\sqrt{1 - \frac{n^2(n^2+1)(\xi_1+\xi_2)^2}{2}} + \sqrt{1 - \frac{n^2(n^2+1)(\xi_1-\xi_2)^2}{2}} - 2\right] \quad (11)$$

$$h_2 = h_2(\xi_1, \xi_2) = \frac{1}{\sqrt{2}}\left[\sqrt{1 - \frac{n^2(n^2+1)(\xi_1+\xi_2)^2}{2}} - \sqrt{1 - \frac{n^2(n^2+1)(\xi_1-\xi_2)^2}{2}}\right], \quad (12)$$

$$f = f(\xi_1, \xi_1', \xi_2, \xi_2', z, z', \theta) = b(1+z)^2[2n^2(\xi_1\xi_1' + \xi_2\xi_2') - (h_1\xi_1'' + h_2\xi_2'')]$$

$$+\frac{bn^2(n^2+1)(1+z)^2}{2}\left(\frac{(\xi_1-\xi_2)(\xi_1'-\xi_2')^2}{h_1-h_2+\sqrt{2}} + \frac{(\xi_1+\xi_2)(\xi_1'+\xi_2')^2}{h_1+h_2+\sqrt{2}}\right)$$

$$+b(1+z)z'[n^2(\xi_1^2+\xi_2^2) - (h_1\xi_1' + h_2\xi_2' + \sqrt{2}\xi_1')] - \sqrt{2}b\xi_1''z(z+2) - (1+2b)zz'$$

$$+b\hat{\mu}_a(1+z)(h_1\xi_1' + h_2\xi_2' + \sqrt{2}\xi_1') - \hat{\mu}_0(1+z) + \hat{\Gamma}_0 + \hat{\Gamma}(\theta). \quad (13)$$

It can be seen that the equations for ξ_1 and ξ_2 are decoupled at linear order.

The natural frequencies ω and mode shapes (ξ_1, ξ_2, z') of the linearized system are:

(1) $\omega = 0$ and $(\xi_1, \xi_2, z') = (0, 0, 1)$. This denotes the rigid-body motion of the disk with the absorbers at their vertices.

(2) $\omega = n$ and $(\xi_1, \xi_2, z') = (0, 1, 0)$. This denotes the case when the absorbers move exactly out-of-phase at equal amplitude and the disk runs at a constant speed.

(3) $\omega = n\sqrt{1+2b}$ and $(\xi_1, \xi_2, z') = (1, 0, \sqrt{2}bn^2)$. This is identical to the only vibration mode of an equivalent single-absorber/disk system, as the absorbers move in exact unison and the disk undergoes torsional oscillations out-of-phase with respect to the absorbers. (Note that this physical interpretation requires that the system be visualized in the (s, θ) co-ordinates.) In designing absorber systems it is generally assumed that the system will operate in this mode, even for moderate and large amplitudes.

An examination of equations (8)–(10) indicates that, due to resonant interactions, bifurcations may occur in ξ_2 if the external torque consists of harmonics of order n or multiple orders of n. The order n harmonic induces a primary resonance of ξ_2, while higher order harmonics of n induce subharmonic resonances of ξ_2 through non-linearities of the system. For the current study, the oscillating part of the torque is restricted to the case of an order n harmonic plus an order $2n$ harmonic, as follows:

$$\hat{\Gamma}(\theta) = \hat{\Gamma}_n \cos(n\theta - \gamma_n) + \hat{\Gamma}_{2n}\cos(2n\theta - \gamma_{2n}). \quad (14)$$

Before turning to the general analysis, there is a special case worthy of note, wherein the $2n$ harmonic of the torque $\hat{\Gamma}(\theta)$ can be totally cancelled by a pair of identical absorbers [6]. For zero absorber damping ($\hat{\mu}_a = 0$), zero order n torque $\hat{\Gamma}_n = 0$, and a balance between the disk damping torque and the constant applied torque ($\hat{\mu}_0 = \hat{\Gamma}_0$), the non-linear system has an *exact* solution given by

$$\xi_1 = 0, \qquad \xi_2 = (1/n)\sqrt{(\hat{\Gamma}_{2n}/bn)}\cos(n\theta - \gamma_{2n}/2 - \pi/4), \qquad z = 0. \quad (15)$$

The frequency of the absorbers' motions in this case is a subharmonic of order two relative to the external torque, since its frequency is n while the external torque is of frequency $2n$. During this motion the absorbers move exactly out-of-phase with respect to each other ($s_1 = -s_2$, since $\xi_1 = 0$), totally cancelling each other's odd order harmonics. However, the order $2n$ harmonics generated by quadratic non-linearities add in such a manner so as to exactly cancel the applied torque, and no other even order harmonics are generated. As a result, the disk will rotate at a constant rate, $z = 0$, over a range of torque amplitudes up to $\hat{\Gamma} = 2bn/(n^2 + 1)$, at which point the absorber masses reach cusps in the epicycloidal paths during their motions. (This limit can be established by determining where the g_i's vanish and the $\mathrm{d}g_i/\mathrm{d}s_i$'s become singular.) The dynamic stability of this response has been derived by Lee *et al*. [6] in the presence of small absorber damping. Those results are contained in the more general analysis that follows.

3. PERTURBATION ANALYSIS

An exact, global solution of the basic system is given in the previous section for the case when $\hat{\mu}_a = 0$ and $\hat{\Gamma}(\theta)$ consists of an order $2n$ harmonic only. However, the corresponding solution for $\hat{\mu}_a \neq 0$ and multi-harmonic $\hat{\Gamma}(\theta)$ is not available in closed form. Therefore, the method of multiple scales (MMS) is used to obtain a second order approximation of this solution for $0 < \hat{\mu}_a \ll 1$ and $0 < \hat{\Gamma}(\theta) \ll 1$. This solution will be used for determining the stability of the unison motion and, more importantly, the post-critical behavior of the system, including an analysis of the performance characteristics of the absorbers in this operating range.

The perturbation analysis is carried out by a newly proposed MMS to the second order, developed by Lee and Lee [11], which is a simplified version of that proposed by Rahman and Burton [12]. The variables and parameters are scaled such that the bifurcation of interest appears at the second non-linear order, i.e., the third order in the scaling parameter, ϵ. They are given by

$$\xi_1 = \epsilon\xi_{1,1} + \epsilon^2\xi_{1,2} + \epsilon^3\xi_{1,3} + \cdots, \qquad \xi_2 = \epsilon\xi_{2,1} + \epsilon^2\xi_{2,2} + \epsilon^3\xi_{2,3} + \cdots, \qquad (16, 17)$$

$$z = \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + \cdots, \qquad \frac{\mathrm{d}}{\mathrm{d}\theta} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 + \cdots, \qquad (18, 19)$$

$$\hat{\mu}_a = \epsilon\mu_a, \qquad \hat{\mu}_0 = \epsilon\mu_0, \qquad \hat{\Gamma}_n = \epsilon\Gamma_n, \qquad \hat{\Gamma}_{2n} = \epsilon^2\Gamma_{2n}, \qquad \hat{\Gamma}_0 = \epsilon\Gamma_{0,1} + \epsilon^2\Gamma_{0,2} + \cdots, \quad (20)$$

where

$$D_p = \partial/\partial\Theta_p, \qquad \Theta_p = \epsilon^p\theta, \qquad p = 0, 1, 2, 3, \ldots.$$

Note that it is assumed that the higher harmonic torque is taken to be smaller in scale than the leading order (n) torque. This is chosen such that the bifurcations caused by the order n and order $2n$ excitations appear simultaneously at the second non-linear order. This is also consistent with applications, as the fundamental order (n) is usually the dominant excitation term.

Details of the analysis are not given here, but some highlights are provided in Appendix A. It is assumed that bifurcations occur only in the variable ξ_2, as is known to be the case when only order $2n$ excitation is present [6]. This assumption is reasonable when the excitation also contains higher harmonics of n since the rigid-body mode of the system and the unison mode of the absorber motion have, respectively, natural frequencies of $\omega = 0$ and $\omega = n\sqrt{1 + 2b}$, which are generally not commensurate with n. The variables z and ξ_1 associated with these two modes are expected to be stable and the homogeneous terms associated with them must vanish. Therefore, only particular terms of their solutions are considered in the derivations. Then the steady-state solutions of the system are carried out to the second non-linear order by using the version of MMS offered by Lee and Lee [11].

By assuming that bifurcations occur only for ξ_2, and balancing the nominal rotation friction with $\hat{\Gamma}_0$, the second order steady state solutions are given by the following:

$$\hat{\Gamma}_0 = \hat{\mu}_0, \tag{21}$$

$$\xi_1 = -\beta_3 \hat{\Gamma}_n \cos(n\theta - \beta_4) - \frac{\sqrt{2}[\beta_1 \sin(2n\theta - \beta_2) + 2b^2 n^4 a_2^2 \sin(2n\theta - 2\phi_2)]}{2bn^3(3-2b)}, \tag{22}$$

$$\xi_2 = a_2 \cos(n\theta - \phi_2), \tag{23}$$

$$z = \frac{\hat{\mu}_a \hat{\Gamma}_n \cos(n\theta - \gamma_n)}{2bn^2} + \frac{3[\beta_1 \cos(2n\theta - \beta_2) + 2b^2 n^4 a_2^2 \cos(2n\theta - 2\phi_2)]}{4bn^2(3-2b)}, \tag{24}$$

where the first condition is simply a balance of the constant applied torque (including load) and the damping torque at the bearing on which the disk rotates, a_2 is the amplitude of the ξ_2 mode, and where the other terms have been defined as follows:

$$\beta_1 = \sqrt{\hat{\Gamma}_n^4 + 4b^2 n^2 \hat{\Gamma}_{2n}^2 + 4bn \hat{\Gamma}_n^2 \hat{\Gamma}_{2n} \sin(2\gamma_n - \gamma_{2n})}, \tag{25}$$

$$\beta_2 = 2\gamma_n + \arg(\hat{\Gamma}_n^2 + 2bn\hat{\Gamma}_{2n} \sin(2\gamma_n - \gamma_{2n}) + 2\mathrm{j}bn\hat{\Gamma}_{2n} \cos(2\gamma_n - \gamma_{2n})), \tag{26}$$

$$\beta_3 = \sqrt{2[4b^2n^2 + (1+4b)^2 \hat{\mu}_a^2)]/4b^2n^3}, \qquad \beta_4 = \gamma_n + \arg(2bn + \mathrm{j}(1+4b)\hat{\mu}_a). \tag{27, 28}$$

(Note that the results become singular when $b = 3/2$. In this case the natural frequencies of the unison and out-of-phase modes are in a 1:2 ratio, an internal resonance condition that will no doubt give rise to other non-linear behavior. However, b is the ratio of absorber inertia to disk inertia, and it will never be larger than one for a tuned absorber system.)

The angular acceleration of the disk, α, is the important variable for assessing absorber effectiveness, and it is equal to $(1+z)z'$. Its approximation to order ϵ^2 is given by

$$\alpha \approx z' = -\frac{\hat{\mu}_a \hat{\Gamma}_n \sin(n\theta - \gamma_n)}{2bn} - \frac{3[\beta_1 \sin(2n\theta - \beta_2) + 2b^2 n^4 a_2^2 \sin(2n\theta - 2\phi_2)]}{2bn(3-2b)}. \tag{29}$$

It can be seen that the order n harmonic in α, which arises from the effect of absorber damping, is not affected by the bifurcation of ξ_2, while the order $2n$ harmonic is strongly influenced by the bifurcation through a_2.

To carry out the bifurcation analysis, it is useful to define the complex response variable $A_2 = \frac{1}{2} a_2 \,\mathrm{e}^{-\mathrm{j}\phi_2}$, where $\xi_2 = 1/2(A_2 + \bar{A}_2)$. (An overbar denotes complex conjugation.) In terms of A_2 it is determined that the annulment of secular terms at second order in the MMS procedure requires that the following condition holds:

$$2\mathrm{j}n \frac{\mathrm{d}A_2}{\mathrm{d}\theta} = -\mathrm{j}n\hat{\mu}_a A_2 - \frac{3(\beta_1 \cos\beta_2 - \mathrm{j}\beta_1 \sin\beta_2 + 8b^2 n^4 A_2^2)\bar{A}_2}{4b(3-2b)}. \tag{30}$$

Two solutions of this condition are now examined in terms of their stability and the attendant absorber performance.

There are two solutions of A_2 in equation (30), the trivial solution $a_2 = 0$, and a non-trivial solution of amplitude

$$a_2^* = \sqrt[4]{9\beta_1^2 - 16b^2 n^2 \hat{\mu}_a^2 (3-2b)^2} / \sqrt{6}bn^2. \tag{31}$$

The phase angle associated with the latter solution is

$$\phi_2 = [\pi + \beta_2 + \arg(3bn^3 a_2^2 + 2\mathrm{j}\hat{\mu}_a(3-2b))]/2. \tag{32}$$

The solution $a_2 = 0$ is stable for $\beta_1 < \beta_1^*$, where the bifurcation parameter condition is

$$\beta_1^* = 4bn\hat{\mu}_a(3 - 2b)/3. \tag{33}$$

The solution corresponding to $a_2 = 0$ becomes unstable at $\beta_1 = \beta_1^*$, at which point the solution $a_2 = a_2^*$ appears, and this solution is stable when it exists.

For $a_2 = 0$, the system response is expressed in equations (22), (23), and (29), with $a_2 = 0$. The two absorbers move in unison, $s_1 = s_2$, since $\xi_2 = 0$, and this is the base operating condition that is generally assumed to occur. The angular acceleration of this response has an order n harmonic whose source is absorber damping and an order $2n$ harmonic arising from a combination of the order $2n$ applied torque and the non-linear effects of the absorber motion, as seen from equation (29) and the term β_1 in equation (25).

The system response corresponding to $a_2 = a_2^*$ can be expressed as

$$\xi_1 = -\beta_3\hat{\Gamma}_n \cos(n\theta - \beta_4) + [2\sqrt{2}\hat{\mu}_a \cos(2n\theta - 2\phi_2)]/3n^2, \tag{34}$$

$$\alpha = -[\hat{\mu}_a\hat{\Gamma}_n \sin(n\theta - \gamma_n)]/2bn + 2\hat{\mu}_a \cos(2n\theta - 2\phi_2), \tag{35}$$

with ξ_2 given in equation (23). The important feature of this response is that the order $2n$ harmonic of α is independent of the external torque, thus rendering this harmonic "saturated" at a fixed value, $2\hat{\mu}_a$. Also, the order $2n$ harmonic of ξ_1 saturates at a level of $2\sqrt{2}\hat{\mu}_a/3n^2$. It should be noted that the acceleration α vanishes as $\hat{\mu}_a \to 0$, which implies that *a pair of order n epicycloidal CPVA's is capable of cancelling a torque consisting of both order n and order* $2n$ *harmonics*. However, some higher harmonic order of α will persist, and may even be amplified, but these effects are not captured by the current analysis.

4. CASE STUDIES

The system response to several combinations of excitation terms is considered. Two very special operating conditions are first briefly described. Then three other cases follow in which β_1 is varied. Finally, the case wherein only the order n harmonic is present in the torque is discussed in some detail. The goal in the latter case is to determine the effects of the bifurcation on the system performance when concentrating on a single harmonic. Where appropriate, analytical results are verified against simulations. The system parameter values used are given by: $b = 0{\cdot}0831$, $n = 2$, $\hat{\mu}_0 = 0{\cdot}05$, and $\hat{\mu} = 0{\cdot}005$; these data are based on an in-line, four-cylinder automotive engine [5].

Two special solutions are first addressed: the first is the case $\hat{\Gamma}_n = 0$, and the second is for $\hat{\Gamma}_n^2 = 2bn\hat{\Gamma}_{2n}$ and $2\gamma_n - \gamma_{2n} = -\pi/2$. The first case has been studied extensively by Lee *et al*. [6], and the system behaves exactly as predicted by the perturbation analysis. This is a special case when the relative phase between the two absorbers is π in the post-bifurcation range, rendering a subharmonic vibration absorber system that has been shown to be very effective in handling a single harmonic torque. The second case represents the situation in which $\beta_1 = 0$ even though the values of $\hat{\Gamma}_n$ and $\hat{\Gamma}_{2n}$ do not vanish. In this case the unison motion of the absorbers is predicted to be stable over the entire torque range, and this has been verified by numerical simulations.

The three cases, in which β_1 is varied, are now presented. They are chosen such that $\hat{\Gamma}_{2n} = 0$ for Case 1, $\hat{\Gamma}_n^2 = 2bn\hat{\Gamma}_{2n}$ and $2\gamma_n - \gamma_{2n} = \pi/2$ for Case 2, and $\hat{\Gamma}_n^2 = 2bn\hat{\Gamma}_{2n}$ and $2\gamma_n - \gamma_{2n} = 0$ for Case 3. Case 1 represents a situation when the external torque consists of only an order n harmonic. Case 2 and Case 3 are chosen to study the effect of the different phases between the order n and order $2n$ harmonics of the external torque for a particular proportion of torque harmonic amplitudes. The β_1 value is varied in each case

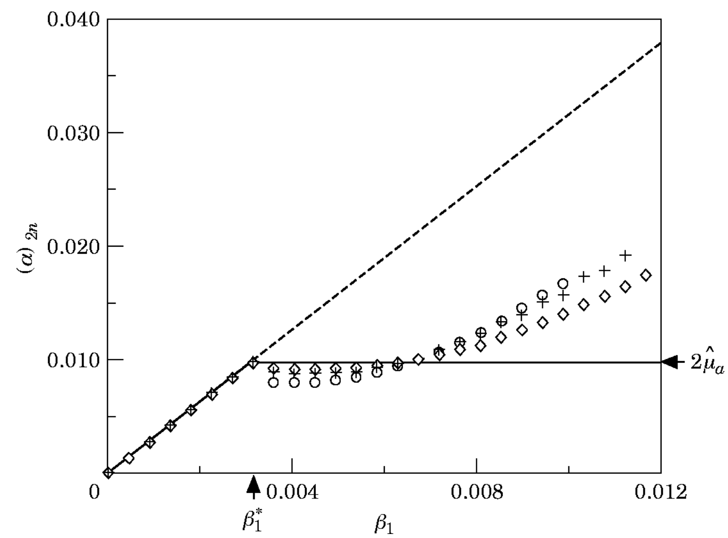

Figure 2. $(\alpha)_{2n}$ versus β_1 from stable and unstable branches of the perturbation result and numerical results of three forcing conditions: ———, unstable; ——, stable; ◇, Case 1; ○, Case 2; +, Case 3.

through the torque amplitudes, maintaining the condition $\hat{\Gamma}_n^2 = 2bn\hat{\Gamma}_{2n}$. The results are shown in Figures 2 and 3 for these three cases, where $(\alpha)_{2n}$ represents the amplitude of the order $2n$ harmonic of the angular acceleration α. (Recall that $(\alpha)_n$, the nth order harmonic, varies in an essentially linear manner with respect to Γ_n.) The "unstable" and "stable" branches in Figure 2 are those obtained from the analytical perturbation results. It is found that $(\alpha)_{2n}$ increases linearly with respect to β_1 until $\beta_1 = \beta_1^*$, at which point a bifurcation occurs. While the perturbation results predict the bifurcation point very well, the post-bifurcation level of $(\alpha)_{2n}$ does not saturate at $2\hat{\mu}_a$ as predicted. This is due to higher order non-linear effects not considered in the present analysis. Figure 3 shows a plot of the a_2's, that is, the amplitude of the out-of-phase component, versus β_1. It can be seen

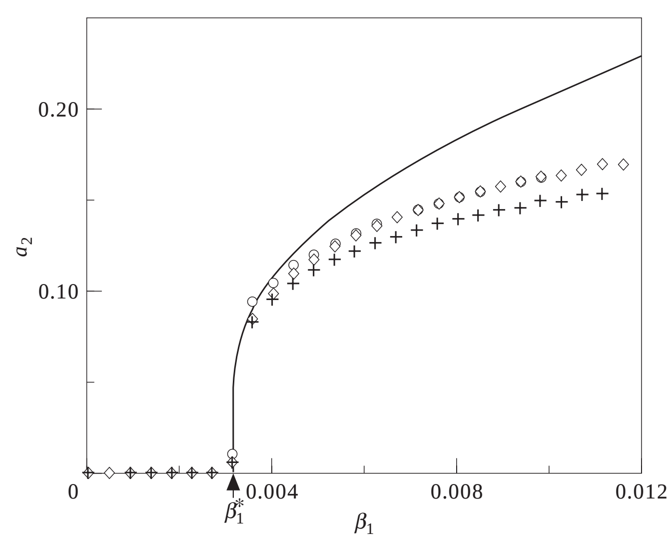

Figure 3. a_2 versus β_1 from perturbation analysis and numerical results of three forcing conditions. ——, Ana; ◇, Case 1; ○, Case 2; +, Case 3.

that the a_2^*'s, i.e., the stable post-bifurcation branches, are somewhat smaller than predicted by the perturbation analysis. Cases 2 and 3 demonstrate that a pair of order n epicycloidal CPVA's are capable of simultaneously reducing two harmonic excitations. Also, it is seen that the general character of the response is the same in these cases as β_1 is varied. However, other response characteristics may occur for other torque proportions or relative phases.

Finally, of particular interest is the case $\hat{\Gamma}_{2n} = 0$, which represents the use of two identical order n epicycloidal CPVA's for an order n external torque. It is desirable to obtain a direct comparison of the performance of a pair of identical absorbers with that of a single absorber wherein the total absorber mass and dissipation are held fixed. The absorbers behave identically in the pre-bifurcation range, but differ in the post-bifurcation range. Here the maximum absorber amplitude can be approximated by neglecting the order $2n$ component of ξ_1 in equation (22) and substiting equations (22) and (23) into equation (6), yielding

$$\max(s) \approx \sqrt{(a_2^2 + \beta_3^2\hat{\Gamma}_n^2 + 2a_2\beta_3\Gamma_n|\cos(\phi_2 - \beta_4)|/2}. \tag{36}$$

In order to survey the effects of the bifurcation on absorber performance, numerical simulations were carried out and the results are shown in Figures 4 and 5. The performance of the paired absorbers is compared with two other configurations with the same total absorber inertia. In the first configuration for comparison, the absorbers are forced to move in unison, as if there were only a single absorber with the same inertia; this is termed the "lumped configuration". In the second configuration, the absorbers are locked at their vertices; this is the baseline operation of the system without any dynamic absorbers.

Figure 4 displays the maximum absorber amplitude max (s) versus $\hat{\Gamma}_n$, where "Ana-1" and "Ana-2" are analytical results from equation (36) with $a_2 = 0$ and a_2 following equation (31), respectively. "Num-1" and "Num-2" are numerical results for the lumped configuration and for the paired absorbers, respectively. In the case with the paired absorbers, the maximum absorber amplitude grows rapidly near the bifurcation torque $\Gamma_n^* = \sqrt{\beta_i^*}$ and then roughly tracks the lumped response in terms of slope thereafter. It is important to note that one absorber reaches its cusp (the peak amplitude physically

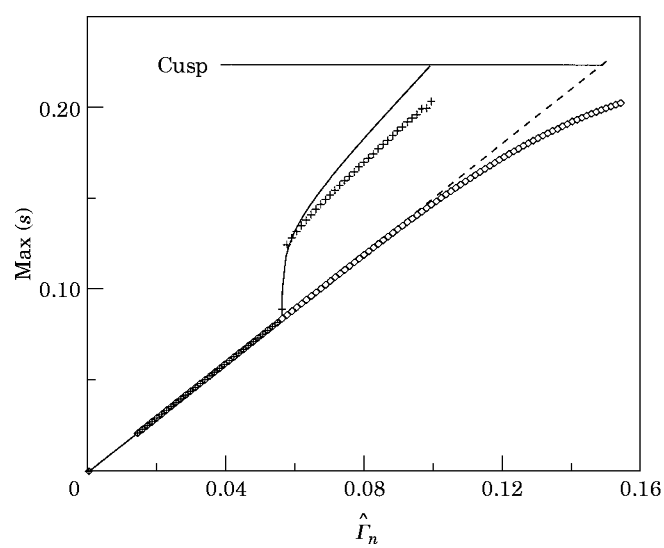

Figure 4. Max (s) versus $\hat{\Gamma}_n$, both of the analytical and numerical results of paired absorbers and the lumped configuration: ———, Ana-1; ——, Ana-2; ◇, Num-1; +, Num-2.

possible) at a torque level smaller than that of the lumped configuration. Figure 5 shows corresponding numerical results for the maximum angular acceleration. Here it is seen that the bifurcation has the advantage of slightly reducing the maximum angular acceleration near the bifurcation.

5. DISCUSSION

The fundamental mechanism of the bifurcation can be explained by considering a special case, as follows. As pointed out by Shaw and Lee [13], the torque generated by an undamped order n epicycloidal CPVA consists of all the odd multiple sine harmonics of order n plus an order $2n$ sine harmonic. (This is determined from the case in which the disk undergoes a constant-rate rotation and the absorber motion is expressed as an order n sine function.) For the special case when $\hat{\mu}_a = 0$ and $\hat{\Gamma}_{2n} = 0$, the paired absorbers in the present situation bifurcate immediately at $\hat{\Gamma}_n = \sqrt{\beta_1^*} = 0$, as seen from equation (33). By substituting equations (23), (25)–(28), (31), and (34) into equation (7), the absorber motions for this case are found to be as follows:

$$s_1 = (\hat{\Gamma}_n/bn^2)\cos(n\theta - \gamma_n - (5/4)\pi), \qquad s_2 = (\hat{\Gamma}_n/bn^2)\cos(n\theta - \gamma_n - (3/4)\pi). \qquad (37)$$

The two absorbers move with the sample amplitude, but with a phase difference of $\pi/2$. It is important to note that the order $2n$ harmonics of the torques generated by these two absorbers cancel each other since the $\pi/2$ phase difference at order n translates to a phase shift of π in the induced order $2n$ torque, thereby resulting in torque cancellation at that order.

In contrast, for the case of the lumped configuration, the absorber amplitude is achieved by setting $a_2 = 0$ in equations (22) and (23), keeping only the order n component, and then substituting the result into equation (7), yielding

$$s_1 = s_2 = (\hat{\Gamma}_n/\sqrt{2}bn^2)\cos(n\theta - \gamma_n - \pi). \qquad (38)$$

By comparing equations (37) and (38), it is seen that the absorber amplitudes are increased by a factor $\sqrt{2}$ in the paired absorber case (this also follows from simple vector addition

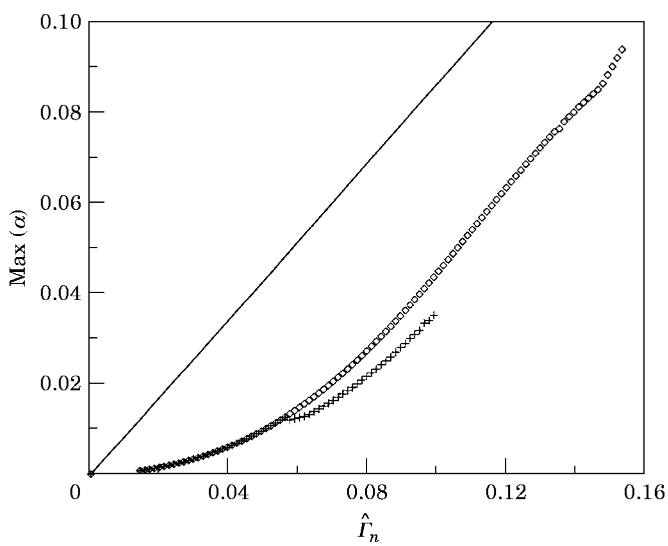

Figure 5. Max (α) versus $\hat{\Gamma}_n$, numerical results of the paired absorbers and the lumped configuration, along with the case with the absorbers locked for a reference: ——, locked; ◇, Num-1; +, Num-2.

of the phase-shifted absorbers's torques). In this way the order n excitation is dealt with equally effectively by both the lumped configuration and the paired configuration, although the latter requires larger absorber amplitudes. However, since the absorbers are not locked in unison, the paired absorbers are able to cancel simultaneously both the order n external excitation and the induced order $2n$ torques by adjusting their amplitudes and phases from the nominal values given above and using non-linear effects of the absorber torques. When an order $2n$ applied torque is also present, the amplitudes and phase shift to accommodate that term in addition.

For a more thorough performance evaluation, higher order harmonics must also be considered, as they are actually amplified in the bifurcated motion. For example, the order $3n$ torque harmonic generated by an absorber grows approximately as s^3. Thus, the amplitude rescaling of $\sqrt{2}$ and the phase shift of $\pi/2$ render the order $3n$ torque twice as large in the paired absorber case when compared with the lumped configuration.

6. CONCLUSIONS

It has been shown that the paired absorbers are capable of simultaneously reducing the two harmonics of the torsional oscillations, but at the expense of increasing higher harmonic amplitudes. In addition, the peak amplitudes of the absorber motions are larger than those for the corresponding unison motion of the lumped configuration in the post-bifurcation range, thereby reducing the applicable torque range. While the perturbation analysis accurately predicts the bifurcation condition, it does not predict the post-bifurcation behavior as well, except in some special cases, due to the effects of higher order harmonics.

An interesting conjecture raised by the results contained herein is that one may find that by using N identical absorbers, the harmonic content of their motions in a post-critical range may adjust to simultaneously counteract N harmonics of the applied torque. If true, multiple absorber systems may have a better than predicted performance, but over a smaller torque range. Recent work has shown that the unison response of a system with N absorbers can indeed bifurcate, and the post-bifurcation dynamics are currently being studied.

As is typical for tuned absorber systems, the level of absorber damping plays a critical role in determining performance, and it is desirable to keep it as small as possible. This is true for designs based on linear or non-linear considerations. An unfortunate reality is that the absorber damping is very difficult to estimate from first principles and is not easily measured, and therefore one must be cautious when sizing absorber systems based on analytical calculations. Experimental testing, of at least the damping levels and how they depend on operating conditions, will be required for actual implementation.

REFERENCES

1. V. J. Borowski, H. H. Denman, D. L. Cronin, S. Shaw, J. P. Hanisko, L. T. Brooks, D. A. Milulec, W. B. Crum and M. P. Anderson 1991 *The Engineering Society for Advanced Mobility Land, Sea, Air and Space, SAE Technical Paper Series* 911876. Reducing vibration of reciprocating engines with crankshaft pendulum vibration absorbers.
2. W. Ker Wilson 1968 *Practical Solution of Torsional Vibration Problems*, volume IV, chapter XXX. London: Chapman and Hall; third edition.
3. M. A. Wachs 1973 *SAE Technical Paper Series* 730894. The main rotor bifilar absorber and its effect on helicopter reliability/maintainability.
4. D. L. Cronin 1992 *Mechanism and Machine Theory* **27**, 517–533. Shake reduction in an automobile engine by means of crankshaft-mounted pendulums.

5. H. H. DENMAN 1992 *Journal of Sound and Vibration* **159**, 251–277. Tautochronic bifilar pendulum torsion absorbers for reciprocating engines.
6. C.-T. LEE, S. W. SHAW and V. T. COPPOLA 1994 In press *ASME Journal of Vibration and Acoustics*. A subharmonic vibration absorber for rotating machinery.
7. L. S. GUNSIOR 1995 *Technical report*, *Department of Mechanical Engineering*, *Michigan State University*. A simulation study of systems with multiple centrifugal pendulum vibration absorbers.
8. C.-P. CHAO, S. W. SHAW and C.-T. LEE 1995 to appear, *ASME Journal of Applied Mechanics*. Stability of the unison response for a rotating system with multiple centrifugal pendulum vibration absorbers.
9. D. E. NEWLAND 1964 *ASME Journal of Engineering for Industry* **86**, 257–263. Nonlinear aspects of the performance of centrifugal pendulum vibration absorbers.
10. M. SHARIF-BAKHTIAR and S. W. SHAW 1992 *ASME Journal of Vibration and Acoustics* **114**, 305–311. Effects of nonlinearities and damping on the dynamics response of a centrifugal pendulum absorber.
11. C. L. LEE and C.-T. LEE 1995 [*preprint*]. A higher order method of multiple scales-periodic steady-state response and stability.
12. Z. RAHMAN and T. D. BURTON 1989 *Journal of Sound and Vibration* **133**, 369–379. On the higher order methods of multiple scales in non-linear oscillators-periodic steady state response.
13. S. W. SHAW and C.-T. LEE 1995 In *Smart Structures*, *Nonlinear Vibration*, *and Control*. Englewood Cliffs, NJ: Prentice Hall. On the nonlinear dynamics of centrifugal pendulum vibration absorbers.

APPENDIX A: LEE AND LEE'S VERSION OF MMS

This appendix highlights the application of Lee and Lee's version of MMS to achieve the steady-state periodic solutions and their stabilities for the system under investigation. Since bifurcations are assumed not to be associated with ξ_1 and y, the homogeneous solutions of ξ_1 and y are neglected, while the steady-state solutions for ξ_2, denoted by $\xi_{2,0}$ are written in the form

$$\xi_{2,0} = B_2\,\mathrm{e}^{\mathrm{j}n\Theta_0} + cc, \tag{A.1}$$

where B_2 is treated as a complex constant when calculating secular terms at higher orders, and $A_2 = \epsilon B_2$ in equation (30). The secular terms of ξ_2 at $O(\epsilon^2)$ and $O(\epsilon^3)$ are

$$-\mathrm{j}n\mu_a B_2 \quad \text{and} \quad -(3(\delta_1 \cos\beta_2 - \mathrm{j}\delta_1 \sin\beta_2 + 8b^2n^4B_2^2)\bar{B}_2)/(4b(3-2b)), \tag{A.2}$$

respectively, where $\beta_1 = \epsilon^2\delta_1$ in equation (25). These two secular terms are recombined by multiplying them by ϵ^2 and ϵ^3, respectively, adding up the result, and recovering the parameters and variables to their original forms. These results are found in the right hand side of equation (30). The left hand side of equation (30) is a recombined result from $2D_0D_i\xi_2$, which is a term obtained from the expansion of $\mathrm{d}^2(\cdot)/\mathrm{d}\theta^2$. These two parts determine the steady-state solutions and their stabilities.

This method is simpler than Rahman and Burton's version of MMS in two ways: system parameters do not need to be expanded in series; and several time derivative terms can be neglected beforehand, since it is known *a priori* that they will be eliminated in the reconsitution procedure, thus saving calculation effort. The method of Lee and Lee simplifies the calculations involved in Rahman and Burton's version of MMS, but the results from the two procedures are identical.